Fractional Optimization Model for Infrared and Visible Image Fusion

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1 Fractional Optimization Model

The fractional optimization model of pre-fused image f_{pre} generated according to infrared image f_{ir} and visible image f_{vis} is as follows:

$$arg\min_{f} \frac{1}{2}\|f-f_{ir}\|_F^2+\frac{\mu}{2}\|D^{v_f}f-D^{v_{f_{ir}}}f_{ir}-D^{v_{f_{vis}}}f_{vis}\|_F^2. \tag{1}$$

where $\|\cdot\|_F$ represents the Frobenius norm, $D^v s$ denotes the v-order discrete fractional gradient of $s\in\mathbb{R}^{n\times n}$ [3].

$$D^v s=\begin{pmatrix}Ms\\ sM\end{pmatrix}\in\mathbb{R}^{2n\times n}, M=\begin{pmatrix}2c_1^v & c & c_3^v & \cdots & c_n^v\\ c & 2c_1^v & \ddots & \ddots & \vdots\\ c_3^v & \ddots & \ddots & \ddots & c_3^v\\ \vdots & \ddots & \ddots & 2c_1^v & c\\ c_n^v & \cdots & c_3^v & c & 2c_1^v\end{pmatrix}\in\mathbb{R}^{n\times n}, \tag{2}$$

and

$$c_k^v=(-1)^k\binom{v}{k}, c_0^v=1, c=c_0^v+c_2^v, (D^v)^T\begin{pmatrix}s_1\\ s_2\end{pmatrix}=Ms_1+s_2M. \tag{3}$$

Let

$$
\begin{aligned}
\varphi_1(f) &= \frac{1}{2}\|f-f_{ir}\|_F^2 = \frac{1}{2}Tr[(f-f_{ir})^T(f-f_{ir})],\\
\varphi_2(f) &= \frac{\mu}{2}\|D^{v_f}f - D^{v_{f_{ir}}}f_{ir} - D^{v_{f_{vis}}}f_{vis}\|_F^2\\
&= \frac{\mu}{2}Tr[(D^{v_f}f - D^{v_{f_{ir}}}f_{ir} - D^{v_{f_{vis}}}f_{vis})^T(D^{v_f}f - D^{v_{f_{ir}}}f_{ir} - D^{v_{f_{vis}}}f_{vis})]\\
&= \frac{\mu}{2}Tr[\begin{pmatrix} M_f f - M_{ir}f_{ir} - M_{vis}f_{vis}\\ fM_f - f_{ir}M_{ir} - f_{vis}M_{vis}\end{pmatrix}^T \begin{pmatrix} M_f f - M_{ir}f_{ir} - M_{vis}f_{vis}\\ fM_f - f_{ir}M_{ir} - f_{vis}M_{vis}\end{pmatrix}]\\
&= \frac{\mu}{2}Tr[(M_f f - M_{ir}f_{ir} - M_{vis}f_{vis})^T(M_f f - M_{ir}f_{ir} - M_{vis}f_{vis})]\\
&+ \frac{\mu}{2}Tr[fM_f - f_{ir}M_{ir} - f_{vis}M_{vis})^T(fM_f - f_{ir}M_{ir} - f_{vis}M_{vis})],
\end{aligned}
\tag{4}
$$

where Tr denotes the trace of the matrix. $\varphi_1(f)$ and $\varphi_2(f)$ are convex functions. The proof is as follows.

Given the function $\Phi: R_1 \to R_2$ is considered as a convex function, then it satisfies: $\forall(x_1,x_2,\tau)\in R_1\times R_2\times[0,1], s.t.\Phi[(1-\tau)x_1+\tau x_2] \le (1-\tau)\Phi(x_1)+\tau\Phi(x_2)$.

Proof 1. Let f_1 and f_2 are two points in the domain of $\varphi_1(f)$, we can obtain that:

$$
\varphi_1(f_1) = \frac{1}{2}Tr[(f_1-f_{ir})^T(f_1-f_{ir})] = \frac{1}{2}Tr(f_1^Tf_1) - Tr(f_1^Tf_{ir}) + \frac{1}{2}Tr(f_{ir}^Tf_{ir}),
\tag{5}
$$

$$
\varphi_1(f_2) = \frac{1}{2}Tr[(f_2-f_{ir})^T(f_2-f_{ir})] = \frac{1}{2}Tr(f_2^Tf_2) - Tr(f_2^Tf_{ir}) + \frac{1}{2}Tr(f_{ir}^Tf_{ir}),
\tag{6}
$$

$$
\begin{aligned}
(1-\tau)\varphi_1(f_1)+\tau\varphi_1(f_2) = \frac{1-\tau}{2}Tr(f_1^Tf_1) + \frac{\tau}{2}Tr(f_2^Tf_2) + \frac{1}{2}Tr(f_{ir}^Tf_{ir})\\
-(1-\tau)Tr(f_1^Tf_{ir}) - \tau Tr(f_2^Tf_{ir}),
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\varphi_1[(1-\tau)f_1+\tau f_2] &= \frac{1}{2}Tr\{[(1-\tau)f_1+\tau f_2-f_{ir}]^T[(1-\tau)f_1+\tau f_2-f_{ir}]\}\\
&= \frac{(1-\tau)^2}{2}Tr((f_1^Tf_1) + \frac{\tau^2}{2}Tr((f_2^Tf_2) + \frac{1}{2}Tr((f_{ir}^Tf_{ir})\\
&+ \tau(1-\tau)Tr((f_1^Tf_2) - (1-\tau)Tr((f_1^Tf_{ir}) - \tau Tr(f_2^Tf_{ir}).
\end{aligned}
\tag{8}
$$

Based on Eq.(7) and Eq.(8), we can obtain that:

$$
\begin{aligned}
(1-\tau)\varphi_1(f_1)+\tau\varphi_1(f_2)-\varphi_1[(1-\tau)f_1+\tau f_2] &= \frac{\tau(1-\tau)}{2}Tr(f_1^Tf_1) + \frac{\tau(1-\tau)}{2}Tr(f_2^Tf_2)\\
&- \tau(1-\tau)Tr((f_1^Tf_2)\\
&= \frac{\tau(1-\tau)}{2}Tr[(f_1-f_2)^T((f_1-f_2)]\\
&= \frac{\tau(1-\tau)}{2}\|f_1-f_2\|_F^2 \ge 0,
\end{aligned}
\tag{9}
$$

that is, $\varphi_1[(1-\tau)f_1+\tau f_2] \le (1-\tau)\varphi_1(f_1)+\tau\varphi_1(f_2)$, which means that $\varphi_1(f)$ is convex.

Proof 2. Let f_1 and f_2 are two points in the domain of $\varphi_2(f)$, we can obtain that:

$$
\begin{aligned}
\varphi_2(f_1) &= \frac{\mu}{2} Tr[(M_f f_1 - M_{ir} f_{ir} - M_{vis} f_{vis})^T (M_f f_1 - M_{ir} f_{ir} - M_{vis} f_{vis})] \\
&+ \frac{\mu}{2} Tr[f_1 M_f - f_{ir} M_{ir} - f_{vis} M_{vis})^T (f_1 M_f - f_{ir} M_{ir} - f_{vis} M_{vis})] \\
&= \frac{\mu}{2} Tr(f_1^T M_f^T M_f f_1) + \frac{\mu}{2} Tr(f_{ir}^T M_{ir}^T M_{ir} f_{ir}) + \frac{\mu}{2} Tr(f_{vis}^T M_{vis}^T M_{vis} f_{vis}) \\
&- \mu Tr(f_1^T M_f^T M_{ir} f_{ir}) - \mu Tr(f_1^T M_f^T M_{vis} f_{vis}) - \mu Tr(f_{ir}^T M_{ir}^T M_{vis} f_{vis}) \\
&+ \frac{\mu}{2} Tr(M_f^T f_1^T f_1 M_f) + \frac{\mu}{2} Tr(M_{ir}^T f_{ir}^T f_{ir} M_{ir}) + \frac{\mu}{2} Tr(M_{vis}^T f_{vis}^T f_{vis} M_{vis}) \\
&- \mu Tr(M_f^T f_1^T f_{ir} M_{ir}) - \mu Tr(M_f^T f_1^T v f_{vis} M_{vis}) - \mu Tr(M_{ir}^T f_{ir}^T f_{vis} M_{vis}),
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\varphi_2(f_2) &= \frac{\mu}{2} Tr(f_2^T M_f^T M_f f_2) + \frac{\mu}{2} Tr(f_{ir}^T M_{ir}^T M_{ir} f_{ir}) + \frac{\mu}{2} Tr(f_{vis}^T M_{vis}^T M_{vis} f_{vis}) \\
&- \mu Tr(f_2^T M_f^T M_{ir} f_{ir}) - \mu Tr(f_2^T M_f^T M_{vis} f_{vis}) - \mu Tr(f_{ir}^T M_{ir}^T M_{vis} f_{vis}) \\
&+ \frac{\mu}{2} Tr(M_f^T f_2^T f_2 M_f) + \frac{\mu}{2} Tr(M_{ir}^T f_{ir}^T f_{ir} M_{ir}) + \frac{\mu}{2} Tr(M_{vis}^T f_{vis}^T f_{vis} M_{vis}) \\
&- \mu Tr(M_f^T f_2^T f_{ir} M_{ir}) - \mu Tr(M_f^T f_2^T v f_{vis} M_{vis}) - \mu Tr(M_{ir}^T f_{ir}^T f_{vis} M_{vis}),
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\varphi_2[(1-\tau) f_1 + \tau f_2] &= \frac{\mu(1-\tau)^2}{2} Tr(f_1^T M_f^T M_f f_1) + \frac{\mu\tau^2}{2} Tr(f_2^T M_f^T M_f f_2) + \frac{\mu}{2} Tr(f_{ir}^T M_{ir}^T M_{ir} f_{ir}) \\
&+ \frac{\mu}{2} Tr(f_{vis}^T M_{vis}^T M_{vis} f_{vis}) + \mu\tau(1-\tau) Tr(f_1^T M_f^T M_f f_2) - \mu(1-\tau) Tr(f_1^T M_f^T M_{ir} f_{ir}) \\
&- \mu(1-\tau) Tr(f_1^T M_f^T M_{vis} f_{vis}) - \mu\tau Tr(f_2^T M_f^T M_{ir} f_{ir}) - \mu\tau Tr(f_2^T M_f^T M_{vis} f_{vis}) \\
&+ \frac{\mu(1-\tau)^2}{2} Tr(M_f^T f_1^T f_1 M_f) + \frac{\mu\tau^2}{2} Tr(M_f^T f_2^T f_2 M_f) + \frac{\mu}{2} Tr(M_{ir}^T f_{ir}^T f_{ir} M_{ir}) \\
&+ \frac{\mu}{2} Tr(M_{vis}^T f_{vis}^T f_{vis} M_{vis}) + \mu\tau(1-\tau) Tr(M_f^T f_1^T f_2 M_f) - \mu(1-\tau) Tr(M_f^T f_1^T f_{ir} M_{ir}) \\
&- \mu(1-\tau) Tr(M_f^T f_1^T f_{vis} M_{vis}) - \mu\tau Tr(M_f^T f_2^T f_{ir} M_{ir}) - \mu\tau Tr(M_f^T f_2^T f_{vis} M_{vis}).
\end{aligned} \tag{12}
$$

Based on Eqs.(10-12), we can obtain that:

$$
\begin{aligned}
(1-\tau)\varphi_2(f_1) + \tau\varphi_2(f_2) - \varphi_2[(1-\tau) f_1 + \tau f_2] &= \frac{\mu\tau(1-\tau)}{2} Tr(f_1^T M_f^T M_f f_1) + \frac{\mu\tau(1-\tau)}{2} Tr(f_2^T M_f^T M_f f_2) \\
&+ \frac{\mu\tau(1-\tau)}{2} Tr(M_f^T f_1^T f_1 M_f) + \frac{\mu\tau(1-\tau)}{2} Tr(M_f^T f_2^T f_2 M_f) \\
&= \frac{\mu\tau(1-\tau)}{2} (\|M_f f_1\|_F^2 + \|M_f f_2\|_F^2 + \|f_1 M_f\|_F^2 + \|f_2 M_f\|_F^2) \geq 0,
\end{aligned} \tag{13}
$$

that is, $\varphi_2[(1-\tau) f_1 + \tau f_2] \leq (1-\tau)\varphi_2(f_1) + \tau\varphi_2(f_2)$, which means that $\varphi_2(f)$ is convex.

According to the properties of convex functions, the sum of two convex functions in the same domain is still a convex function [2], and the local minimum of the convex function is the global minimum [4]. Therefore, we can find the global minimum of Eq.(1). Under the first-order condition of Eq.(1), we can obtain that:

$$
f - f_{ir} + \mu (D^{v_f})^T (D^{v_f} f - D^{v_{f_{ir}}} f_{ir} - D^{v_{f_{vis}}} f_{vis}) = 0, \tag{14}
$$

then based on Eq.(2) and Eq.(3), we can obtain that:

$$
f + \mu (M_{v_f}^2 f + f M_{v_f}^2) = f_{ir} + \mu (D^{v_f})^T (D^{v_{f_{ir}}} f_{ir}) + \mu (D^{v_f})^T (D^{v_{f_{vis}}} f_{vis}), \tag{15}
$$

and

$$(E+\mu M_{v_f}^2)f+f(\mu M_{v_f}^2)=(E+\mu M_{v_f}M_{v_{f_{ir}}})f_{ir}+f_{ir}(\mu M_{v_{f_{ir}}}M_{v_f}) \\ +f_{vis}(\mu M_{v_{f_{vis}}}M_{v_f})+(\mu M_{v_f}M_{v_{f_{vis}}})f_{vis}, \tag{16}$$

where E denotes the identity matrix. We find that Eq.(16) is equivalent to the form of $AX+XB=C$, namely Sylvester equation. It can be solved by the Bartels-Stewart algorithm [1].

References

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