Fixed Point Layers for Geodesic Morphological Operations

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Introduction

1. Morphological operators by reconstruction = contour preserved operators for Convolutional Neural Networks.
2. Interpretation of its Jacobian matrix in terms if *fixed points* and *basin attraction*
3. Experimental results for *learning geometric attributes* and the generalisation for image denoising in training in *Only one noise level and only one dataset*

Morphological Reconstruction

Let us consider two numerical functions $f, g \in \mathcal{F}(\Omega, \mathbb{R})$, the set of functions mapping from space of points Ω to $\mathbb{R}$, the set of different possible values of the image. Let f, g be such that $f \leq g$, f is called in [Soi13] the *marker* and g the *mask*. The geodesic dilation of size one of f with respect to g is denoted by $\delta_g^{(1)}(f)$ and is defined as the point-wise minimum between g and the elementary dilation $\delta_{\mathtt{SE}}$ in a given local neighbourhood $\mathtt{SE}$

$$\delta^{(1)}(f,g)(x) := \delta_g^{(1)}(f)(x) := \delta_{\mathtt{SE}}(f)(x) \wedge g(x) \quad (1)$$

where $\wedge$ denotes the minimum coordinate-wise operation. The *reconstruction by dilation* of a mask g from a marker f is defined as the geodesic dilation of f with respect to g iterated until stability and is denoted by $R_g^{\delta}(f)$:

$$R^{\delta}(f,g)(x) := R_g^{\delta}(f(x)) := \underbrace{\delta_g^{(1)} \circ \ldots \circ \delta_g^{(1)}}_{k \text{ times}}(f(x)) \quad (2)$$

where k is such that $\delta_g^{(k)}(f) = \delta_g^{(k+1)}(f)$. The reconstruction by dilation extracts the *domes* or *peaks* of the mask which are marked by the marker.

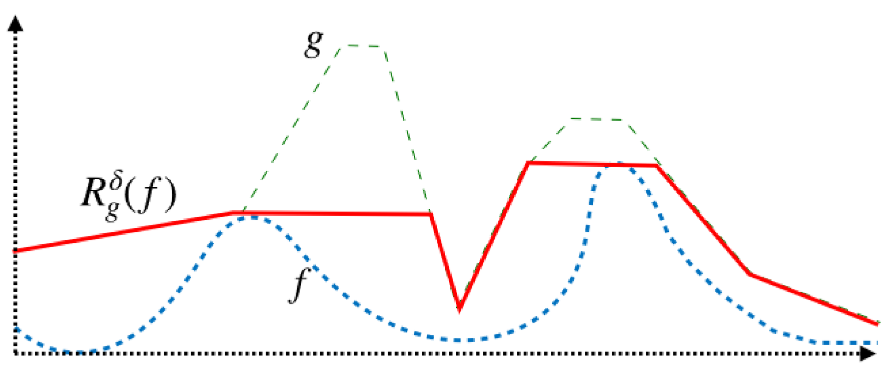

Figure: Reconstruction of mask g from a marker f

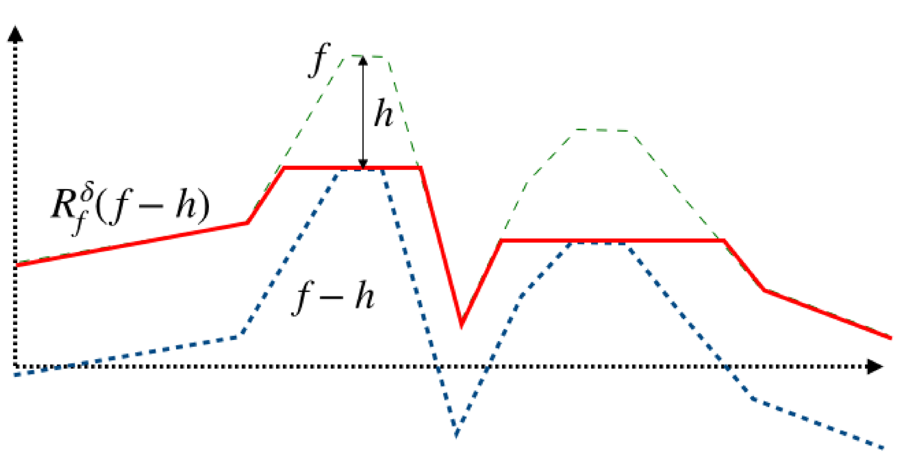

Figure: HMAX transform

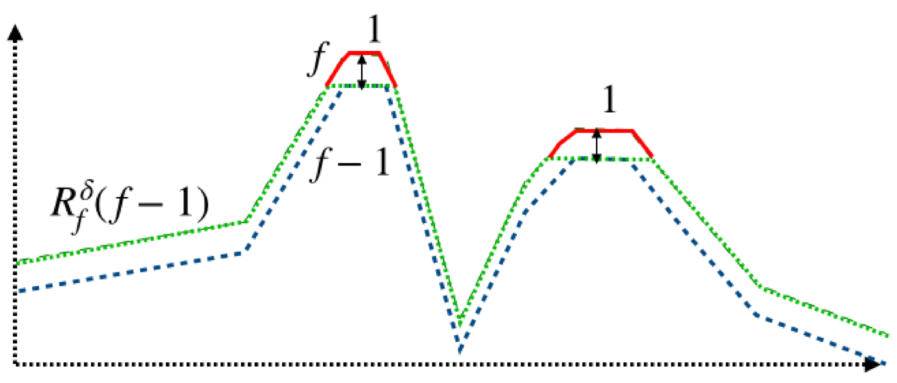

Figure: RMAX transform. The illustrated example use $\epsilon = 1$, but in practical implementation it can be a small number.

Take home message

1. $R_g^{\delta}(f)$ is contour preserving "layer".
2. Reconstruction $R_g^{\delta}(f)$ has no parameter
3. $R_g^{\delta}(f) \leq g \quad \forall f, g$
4. Implementation by Dynamic control flow.
5. Adaptive Reception Field size.
6. Defined for $1D, 2D, \ldots, nD$ images.

Interpretation of Jacobian matrix

For a multivariate, vector-valued function $\tau : \mathbb{R}^n \mapsto \mathbb{R}^n$, the Jacobian is a $n \times n$ matrix denoted by $\mathbf{J}_{\tau}$, containing all first order partial derivatives
The Jacobian matrix of (2) with respect to f is determined by

$$\mathbf{J}_{R^{\delta}(f,g)}(f(x)) = \begin{cases} 1 \text{ in } (i,i) & \text{if } f(x_i) = R^{\delta}(f,g)(x_i) \\ 1 \text{ in } (i,j) & \text{if } x_j \in \mathtt{BA}_{x_i}(\delta_g^{(1)}(f)) \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

and equivalent with respect to the mask g is

$$\mathbf{J}_{R^{\delta}(f,g)}(g(x)) = \begin{cases} 1 \text{ in } (i,i) & \text{if } g(x_i) = R^{\delta}(f,g)(x_i) \\ 1 \text{ in } (i,j) & \text{if } x_j \in \mathtt{BA}_{x_i}(\delta_g^{(1)}(f)) \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

We highlight that the basin of attraction in both (3) and (4) are flat zones, i.e., $x_j \in \mathtt{BA}_{x_i}(\delta_g^{(1)}(f)) \Rightarrow R^{\delta}(f,g)(x_j) = R^{\delta}(f,g)(x_i)$.

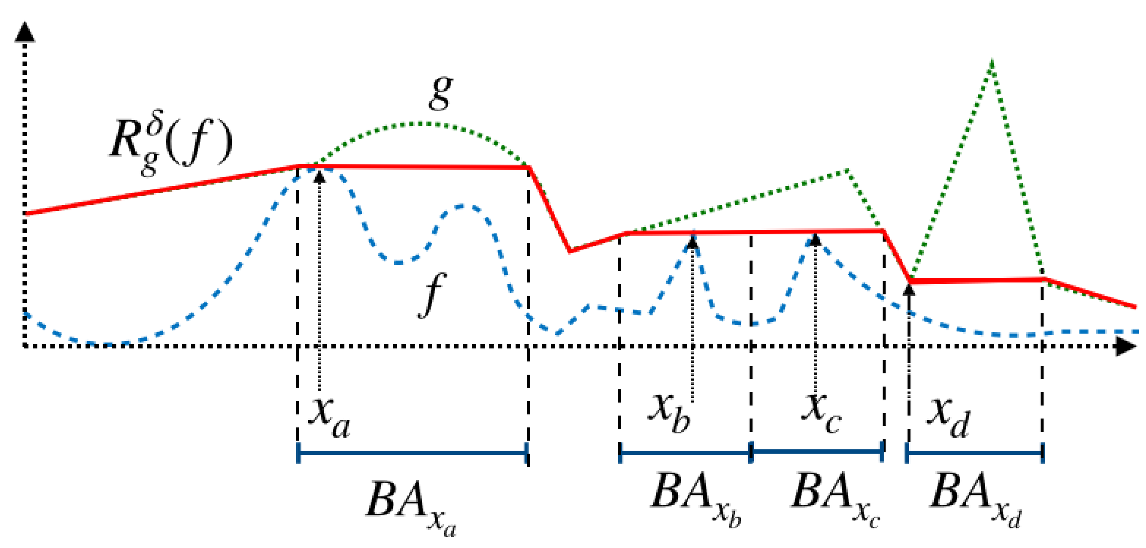

Figure: Basins of attraction with cardinality greater than one. $\mathtt{BA}_{x_a}, \mathtt{BA}_{x_b}$ and $\mathtt{BA}_{x_c}$ contribute to the Jacobian with respect to the mask f in (3) and are associated with a local maxima of f. The $\mathtt{BA}_{x_d}$ contributes to the Jacobian with respect to the marker g (4), and is associated with a local minimum of g.

As a final observation, the number of nonzero values in $\mathbf{J}_{R_g^{\delta}(f)}(f(x)) + \mathbf{J}_{R_g^{\delta}(f)}(g(x))$ is equal to n (number of pixels).

Take home message

1. Jacobian with respect to f: Gradient passes through some maximum of f.
2. Jacobian with respect to g: Gradient passes through some minium of g.

Available Code

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https://github.com/Jacobiano/
GeodesicMorphological
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Learning geometric attributes on simple objects

Each example is a random image with no overlapping objects with random size following an uniform distribution between $[3, 20]$ pixels (the image size is 128×128 pixels). Considered geometrical attributes: Area, Perimeter, Area of Bounding-Box and Eccentricity. 1024 random images are generated for training and 128 for validation. Two models composed of three convolutional layers of kernel size 5×5 with 24 filters with Relu activation functions are trained to predict the value of a geometric attribute, with the difference for the model denoted as $\mathtt{CNN}_R$ a reconstruction by dilation (2) is included of the last layer with the input image used as mask.

Table: Quantitative comparison of Experiment 3.1. The average MSE over ten repetitions in the validation set is reported. $\mathtt{CNN}$ and $\mathtt{CNN}_R$ models have the same number of parameters.

Attribute	$\mathtt{CNN}$	$\mathtt{CNN}_R$	Improvement
Area	0.001084	0.000546	49.61%
Perimeter	0.000683	0.000248	64.36%
Bounding Box Area	0.000504	0.000474	6.08%
Eccentricity	0.003715	0.000301	91.87%

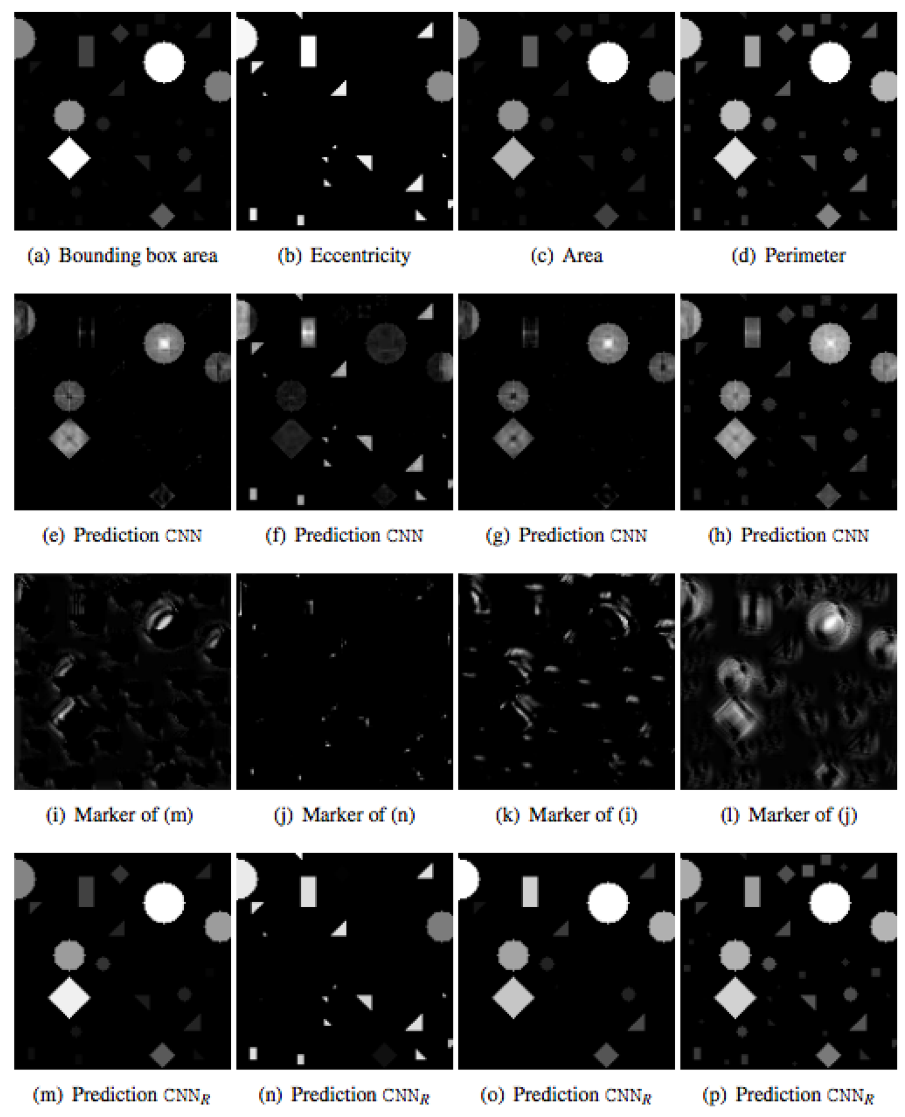

Figure: a) Example of ground truth. b) Bounding box Area c) Eccentricity d) Area e) Perimeter. f) Example of prediction for the attribute perimeter e) for a $\mathtt{CNN}$ in f) and the proposed $\mathtt{CNN}_R$ in g). Both trained models in f) and g) have the same number of parameters. Validation loss in Table. 1

Denoising (Only one noise level and only one database)

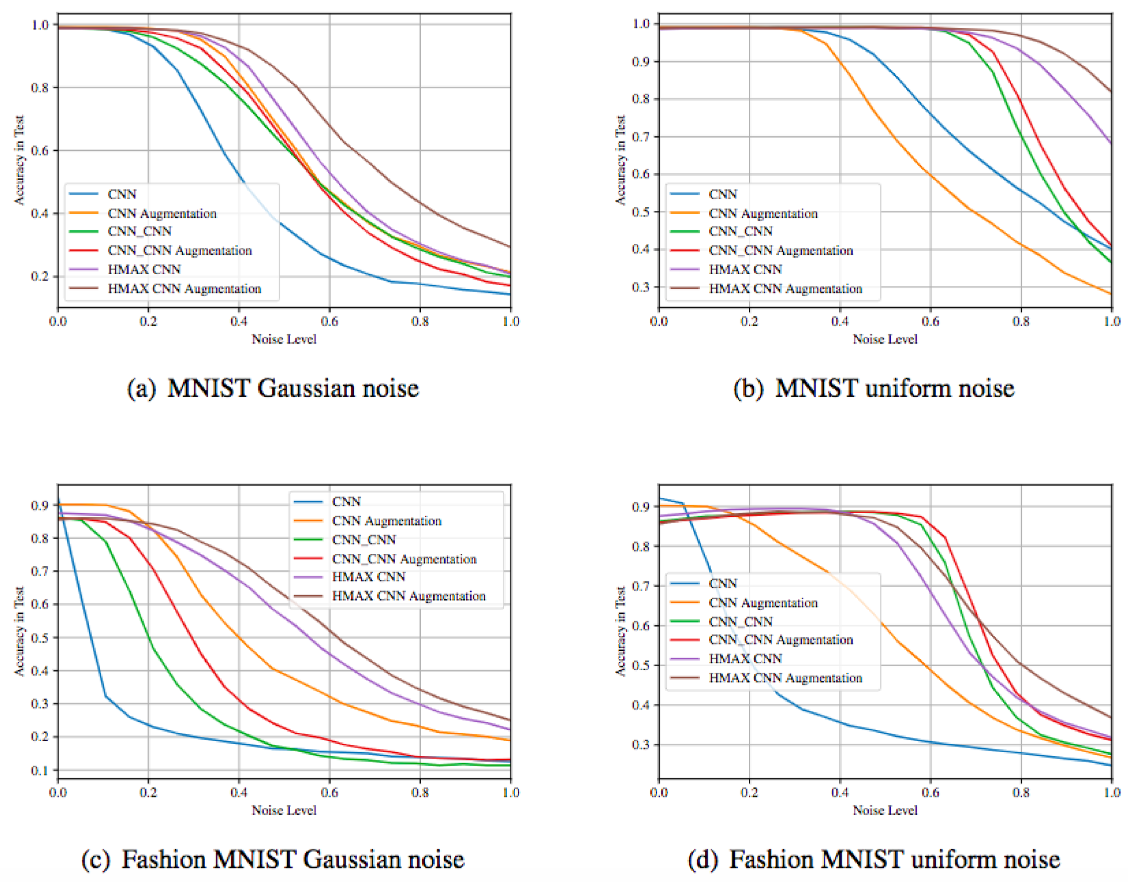

Figure: Classification accuracy for MNIST and Fashion MNIST with additive Gaussian and Uniform noise with $\mu = 0$ and $\sigma \in \{0., 0.05, \ldots, 1\}$. Denoising blocks has been trained *only* on MNIST with additive noise distributed as an absolute value zero-mean Gaussian with $\sigma = 0.1$. Models training with augmentation by additive random Gaussian noise at $\mu = 0$ and σ between 0 and 0.2.