Mathematical morphology provides powerful nonlinear tools that are suitable for visual tasks. However, devising a sequence of operation and their parameters for a given problem is not straightforward and requires expert knowledge. So, efforts have been made to automatically learn their parameters from data. These learnable structures are termed morphological neurons. It has been shown, that the applicability of morphological neurons is not just limited to visual tasks; they are also effective for classification [1] and regression [3]. But the literature lacks their theoretical analysis for practical purposes. So, in this work we have theoretically analysed the properties of morphological neurons and shown that a specific arrangement of the neurons can approximate any continuous function. To be more precise, we have defined a structure called a Morphological Block and shown that a sequence of two morphological blocks can work as a universal approximator. However, to facilitate the theoretical analysis, we have restricted ourselves to the 1D version of the morphological operators, where the operators work over the whole input at once, not locally.

**Morphological Network: How Far Can We Go with Morphological Neurons?**

**Dilation and Erosion Neuron**

Given an input $x \in \mathbb{R}^d$ and a structuring element $s \in \mathbb{R}^d$, the operation of dilation ($\oplus$) and erosion ($\ominus$) neurons are defined respectively as:

$$x \oplus s = \max(x + s),$$

$$x \ominus s = \max(x - s),$$

where $x_i$ denotes $i^{th}$ element of input vector $x$. In these neurons, the structuring element ($s$) is learned in the training phase.

The max and min operators used in the dilation and erosion neurons are only piece wise differentiable. To overcome this problem we propose to use the soft version of max and min to define soft dilation and soft erosion neurons as follows.

$$x \oplus^\beta s = \frac{1}{\beta} \log \left( e^{\beta x} + e^{\beta s} \right),$$

$$x \ominus^\beta s = \frac{1}{\beta} \log \left( e^{\beta x} - e^{\beta s} \right),$$

where $\oplus^\beta$ and $\ominus^\beta$ denote the soft dilation and soft erosion, respectively, and $\beta$ is the "softness" of the soft operations.

**Morphological Block**

Fig 1. Architecture of a single layer morphological block. It contains an input layer, a dilation-erosion layer with $m$ dilation and $m$ erosion neuron and a linear combination layer with $c$ neurons producing the output.

**Equivalence of configurations**

Theorem 1. If we denote $D_m B_n$, as a layer with $m$ dilation neurons and $n$ erosion neurons and $L$ as a linear combination layer, the following may be said about their configurations.

i. The architecture $D_m B_n \rightarrow D_m B_n \rightarrow \cdots \rightarrow D_m B_n$, consisting only of dilation layers, is equivalent to the architecture $D_m B_n \rightarrow B_n \rightarrow L$, with a single dilation layer. A similar statement is true if one considers architectures with only purely erosion layers.

ii. The architecture $D_m B_n \rightarrow D_n B_m$, is not equivalent to $D_n B_m$, Similarly, it is not equivalent to $D_m B_m$, and consequently, the architectures $D_m B_n \rightarrow D_n B_m$ and $D_m B_m \rightarrow D_n B_n$ are not equivalent.

iii. The architecture $D_m B_n \rightarrow D_n B_n \rightarrow L$ is not equivalent to $D_n B_n \rightarrow L$.

Theorem 2 (Universal approximator). Two morphological blocks applied sequentially can approximate continuous functions over arbitrary compact sets.

**Morphological Block: A sum of hinge functions**

Definition 1 ($k$-order Hinge Function [7]): A $k$-order hinge function $H_a(x)$ consists of $(k+1)$ hyperplanes continuously joined together. It may be defined as

$$H_a(x) = \max\{w_0 x + b_0, w_1 x + b_1, \ldots, w_{k+1} x + b_{k+1}\}$$

Proposition 1. The function computed by a Morphological Block (denoted by $M(a)$) with $n$ dilation and $m$ erosion neurons followed by their linear combination, is a sum of multi-order hinge functions.

In fact, we can show that

$$M(a) = \sum_{i=1}^{\min(k,n+m)} a_i H_{i}(x),$$

where $k = m + n$, $a_i \in (-1,1)$ and $H_{i}(x)$. $1 \leq i \leq k$, are $i$-order hinge functions. The proof is given in the supplementary material.

**Results**

Table 1: Accuracy on MNIST and Fashion-MNIST datasets using a single hidden layer with 400 morphological neurons.

<table>
<thead>
<tr>
<th>Architecture</th>
<th>CIFAR10</th>
<th>CIFAR100</th>
<th>SVHN</th>
</tr>
</thead>
<tbody>
<tr>
<td>MNIST (a)</td>
<td>98.33%</td>
<td>98.02%</td>
<td>99.79%</td>
</tr>
<tr>
<td>Fashion-MNIST</td>
<td>89.87%</td>
<td>89.84%</td>
<td>89.70%</td>
</tr>
</tbody>
</table>

Table 2: Test accuracy achieved on CIFAR-10 and SVHN dataset by different networks when the number of neurons ($k$) in the hidden layer is varied. The value of $\beta$ is 12 and 0.5 for CIFAR10 and SVHN respectively.

**Two Morphological Blocks: An universal approximator**

Lemma 1. Any linear combination of hinge functions $\sum_{i=1}^{n} a_i H_{i}(x)$ can be represented over an arbitrary compact set $K$ as a two sequential morphological block consisting of dilation neurons only.

**References**


